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Hence, $t_1 = (2v \sin \varphi)/g$, $t_2 = (2v_1 \sin \varphi_1)/g = (2ev \sin \varphi)/g$, $t_3 = (2v_2 \sin \varphi_2)/g = (2e^2 v \sin \varphi)/g$, etc.

$$T = t_1 + t_2 + t_3 + \cdots = \frac{2v \sin \varphi}{g}[1 + e + e^2 + e^3 + \cdots] = \frac{2v \sin \varphi}{g(1 - e)}.$$

Equating tangential components before and after each impact we obtain

$$\begin{gathered} v_1 \cos \varphi_1 = v \cos \varphi, \qquad v_2 \cos \varphi_2 = v_1 \cos \varphi_1 = v \cos \varphi, \\ v_3 \cos \varphi_3 = v_2 \cos \varphi_2 = v \cos \varphi, \text{ etc.} \end{gathered} \tag{2}$$

From (1) and (2) we have

$$v_1^2 \sin 2\varphi_1 = ev^2 \sin 2\varphi, \qquad v_2^2 \sin 2\varphi_2 = e^2 v^2 \sin 2\varphi, \text{ etc.}$$

Hence Total Range $= R_1 + R_2 + R_3 + \cdots$

$$= \frac{v^2 \sin 2\varphi}{g} + \frac{v_1^2 \sin 2\varphi_1}{g} + \frac{v_2^2 \sin 2\varphi_2}{g} + \cdots$$

$$= \frac{v^2 \sin 2\varphi}{g}[1 + e + e^2 + \cdots] = \frac{v^2 \sin 2\varphi}{g(1 - e)}.$$

Since the plane is smooth the particle will move with a uniform velocity $v \cos \varphi$ after it ceases to rebound.

Also solved by Elijah Swift, L. Sivian, Clifford N. Mills, Horace Olson, A. H. Wilson, and J. W. Clawson.

NUMBER THEORY.

195. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Find triangles whose sides are integers and one of whose angles is 60°.

II. Solution of the generalized problem by R. A. Johnson, Western Reserve University.

It is evident that there is no solution, unless the cosine of the given angle is rational. If, however, this is the case, there are always solutions. Let us assume $\cos \theta = m/n$ where m and n are relatively prime integers, such that $n > 0$, $|m| < n$.

Now if a, b, c be sides of a triangle in which θ is the angle opposite a, we have

$$b^2 - 2\frac{m}{n}bc + c^2 = a^2. \tag{1}$$

Let

$$a = b - \frac{q}{p}c \tag{2}$$

where, as is always possible, p and q have no common factor. Moreover, it is obviously sufficient to consider only *positive* values of p and q; for a case in which they would have unlike signs may be reduced to this case by a change of notation,

i. e., interchanging b and c. Making the substitution and solving, we obtain

$$\frac{c}{b} = \frac{2p(mp - nq)}{n(p^2 - q^2)}\,. \tag{3}$$

Combining (2) and (3),

$$\frac{a}{b} = \frac{n(p^2 + q^2) - 2mpq}{n(p^2 - q^2)}\,. \tag{4}$$

It does not follow, however, that when (4) is true c/b necessarily has the form (3). We find by substituting from (4) in (1) that

$$\frac{c}{b} = \frac{2p(mp - nq)}{n(p^2 - q^2)} \quad \text{or} \quad \frac{2q(np - mq)}{n(p^2 - q^2)} \tag{5}$$

and each of these leads to the value (4) for a/b. Hence these expressions yield the only possible solutions, and we have as necessary conditions for solutions: Either

$$(6_1)\ \begin{cases} a_1 = \dfrac{X}{Y_1}[n(p^2 + q^2) - 2mpq], \\ b_1 = \dfrac{X}{Y_1}[n(p^2 - q^2)], \\ c\ \ = \dfrac{X}{Y_1}[2p(mp - nq)], \end{cases} \quad \text{or} \quad (6_2)\ \begin{cases} a_2 = \dfrac{X}{Y_2}[n(p^2 + q^2) - 2mpq], \\ b_2 = \dfrac{X}{Y_2}[n(p^2 - q^2)], \\ c_2 = \dfrac{X}{Y_2}[2q(np - mq)], \end{cases}$$

where Y_i $(i = 1, 2)$ is the highest common factor of the corresponding set of three expressions in brackets, and X is any positive integer. It remains to determine what limitations must be imposed on p and q in order that these formulas shall actually yield solutions.

The numbers a_1, b_1, c_1 will all be positive, if and only if

$$m > 0, \qquad \frac{q}{p} < \frac{m}{n},$$

and a_2, b_2, c_2 are all positive if and only if

$$q < p.$$

When these conditions are satisfied, we easily show that in either case we have

$$a + b > c, \quad \text{and} \quad b - a < c$$

so that a, b, c are sides of a triangle. Moreover,

$$\cos A \equiv \frac{b^2 + c^2 - a^2}{2bc} = \frac{m}{n}\,,$$

and we actually have a solution.

It will be noted that the expressions in brackets in formulas 6_1 and 6_2 will often have a common factor. In the well-known special case of a right triangle, we need not compute in the case where a common factor exists, for it is known that by a different choice of p and q we get a similar triangle whose sides are prime numbers. Mr. Martin, in the solution above referred to, assumes (apparently correctly) that this is true also when $m/n = \frac{1}{2}$. But in the general case this conclusion is erroneous. That is, if we use the formulas

$$
\begin{aligned}
\bar{a} &= n(p^2 - q^2) - 2mpq, & \bar{b} &= n(p^2 - q^2), \\
\bar{c}_1 &= 2p(mp - nq), & \bar{c}_2 &= 2q(np - mq),
\end{aligned}
\tag{7}
$$

we see that

(a) in general, neither of the sets (a, b, c) has a common factor that could be removed from the expressions;

(b) there exist triangles which cannot be expressed in the form (7) by any choice of p and q subject to the limitations we have determined.

For, adding, we have

$$
\begin{aligned}
\bar{a} + \bar{b} + \bar{c}_1 &= 2p(p - q)(m + n), \\
\bar{a} + \bar{b} + \bar{c}_2 &= 2(p + q)(np - mq),
\end{aligned}
$$

and, for example, if $a + b + c$ be a prime number no choice of p and q will satisfy either one of these alternative necessary conditions. It follows that there are solutions of our problem, sets having no common factor, that must be obtained by deriving from formulas 6_1 and 6_2 sets having common factors, then dividing out such factors.

We summarize as follows:

Let m and n be any two integers without common factor, $n > 0$, $| m | < n$. If a, b, c be three integers without a common factor, representing lengths of the sides of a triangle in which $\cos (b, c) = m/n$, then the following formulas always give sets (a, b, c), and they give all possible sets:

I: (applicable only when $m > 0$)

$$
\left\{
\begin{aligned}
a &= \frac{1}{F}[n(p^2 + q^2) - 2mpq], \\
b &= \frac{1}{F}[n(p^2 - q^2)], \\
c &= \frac{1}{F}[2p(mp - nq)],
\end{aligned}
\right.
\qquad
\left\{
\begin{aligned}
a &= \frac{1}{F}[n(p^2 + q^2) - 2mpq], \\
b &= \frac{1}{F}[n(p^2 - q^2)], \\
c &= \frac{1}{F}[2q(np - mq)],
\end{aligned}
\right.
$$

where p, q are any two relatively prime positive integers, such that in I $q/p < m/n$, in II $q < p$ and in each case F is the highest common factor of the three brackets.

II: (applicable for all values of m)

We can tabulate all possible common factors of either set of three brackets, namely; the following, and no others, will be common factors: in several cases, 2 is a common factor; any factor common to $(m + n)$ and $(p + q)$; any factor common to $(m - n)$ and $(p - q)$; in I, any factor common to n and p; in II, any factor common to n and q.

An excellent solution of the special problem was received from S. A. Joffe.

210. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

If a and b are relatively prime and $(a + b)$ is even, then

$$(a - b)ab(a + b) \equiv 0 \pmod{24}.$$

Solution by L. C. Mathewson, Urbana, Ill.

Since a and b are relatively prime, at most only one can be even; furthermore, since their sum is even, they must both be odd. Accordingly, let

$$a = 2a_1 + 1, \qquad b = 2b_1 + 1.$$

Then

$$\begin{aligned}(a - b)(a + b) &= 4a_1^2 - 4b_1^2 + 4a_1 - 4b_1\\ &= 4(a_1 + b_1 + 1)(a_1 - b_1).\end{aligned}$$

If, now, a_1 and b_1 are both even or both odd, $a_1 - b_1$ is divisible by 2 and thus $(a - b)(a + b)$ divisible by 8. If either a_1 or b_1 is odd (the other even), then $a_1 + b_1 + 1$ is divisible by 2 and thus $(a - b)(a + b)$ divisible by 8. Hence, in either case, under the conditions of the problem

$$(a - b)ab(a + b) \text{ is divisible by } 8.$$

Should at least one of the two given numbers be divisible by 3, then the solution would be evident. If neither a nor b is divisible by 3, then

$$a \equiv 1 \text{ or } 2, \bmod 3, \qquad \text{and} \qquad b \equiv 1 \text{ or } 2, \bmod 3.$$

If both a and b are congruent to 1 or to 2 (mod 3), then $(a - b)$ is divisible by 3; if one is congruent to 1 and the other to 2, then $(a + b)$ is divisible by 3. Hence, in all cases under the conditions of the problem

$$(a - b)ab(a + b) \text{ is divisible by } 3.$$

Therefore, since $(a - b)ab(a + b)$ is divisible by 8 and also by 3,

$$(a - b)ab(a + b) \equiv 0, \pmod{24}.$$

Also solved by S. W. Reaves, H. T. Bigelow, S. A. Joffe, Elijah Swift, Horace Olson, E. E. Whitford, and H. C. Feemster.